Non strict and strict hyperbolic systems for the Einstein equations

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Abstract

The integration of the Einstein equations split into the solution of constraints on an initial space like 3 - manifold, an essentially elliptic system, and a system which will describe the dynamical evolution, modulo a choice of gauge. We prove in this paper that the simplest gauge choice leads to a system which is causal, but hyperbolic non strict in the sense of Leray - Ohya. We review some strictly hyperbolic systems obtained recently.

1   Introduction.

The Einstein equations equate the Ricci tensor of a pseudo riemannian 4-manifold (V, g), of lorentzian signature, the spacetime, with a phenomelogical tensor which describes the sources which we take here to be zero (vacuum case). The Einstein equations are a geometric system, invariant by diffeomorphisms of V and the associated isometries of g. From the analyst point of view they constitute a system of second order quasilinear partial differential equations which is over determined, Cauchy data must satisfy constraints, and underdetermined, the characteristic determinant is identical to zero. An important problem for the study of solutions, their physical interpretation and numerical computation is the choice of a gauge, i.e. a priori hypothesis for instance on coordinates choice, such that the evolution of initial data satisfying the constraints is well posed.

The geometric initial data are a 3-manifold M endowed with a riemaniann metric and a symmetric 2-tensor which will be the extrinsic curvature of M embedded in $V = M \times R$. A natural gauge choice seemed to be the data on V of the time lines by their projection N on R and β on M. Such a choice has been extensively used in numerical computation, though the evolution system $R_{ij} = 0$ obtained with such a choice was not known to be well posed. In this article we will show that the system is indeed hyperbolic in the sense of Leray-Ohya, in the Gevrey class $\gamma = 2$, and is causal, i.e. the domain of dependence of its solutions is determined by the light cone. When the considered evolution system is satisfied the constraints are preserved through a symmetric first order evolution system. Consideration of a system of order 4 obtained previously by combination of the equations $R_{ij} = 0$ with the constraints give the same result.

In section 8 we recall how the old harmonic gauge, interpreted now as conditions on N and β, gives a strictly hyperbolic evolution system and the larger functional spaces where local existence and global geometrical uniqueness of solutions are known.

In recent years, since the paper of C-B and York 1995, there has been a great interest in formulating the evolution part of Eintein equations as a first order symmetric hyperbolic system for geometrically defined unknowns. Several such systems have been devised. Particularly interesting are those constructed with the Weyl tensor (H. Friedrich, see review article 1996) or the Riemann tensor (Anderson, C-B and York 1997) because they lead to estimates of the geometrically defined Bel-Robinson energy used in some global existence proofs (Christodoulou and Klainerman 1989). We recall in the last section the symmetric hyperbolic Einstein-Bianchi system and the corresponding Bel-Robinson energy.

2   Einstein equations.

The spacetime of general relativity is a pseudo riemannian manifold (V, g),of lorentzian signature (- + + +). The Einstein equations link its Ricci tensor with a phenomelogical stress energy tensor which describes the sources. They read

$$Ricci(g) = \rho$$

that is, in local coordinates $x^\lambda, \lambda = 0, 1, 2, 3$, where $g = g_{\lambda\mu} dx^\lambda dx^\mu$,

$$R_{\alpha\beta} \doteq \frac{\partial}{\partial x^\lambda}\Gamma^\lambda_{\alpha\beta} - \frac{\partial}{\partial x^\alpha}\Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\lambda} = \rho_{\alpha\beta}$$

where the $\Gamma's$ are the Christoffel symbols:

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}(\frac{\partial}{\partial x^\alpha}g_{\beta\mu} + \frac{\partial}{\partial x^\beta}g_{\alpha\mu} - \frac{\partial}{\partial x^\mu}g_{\alpha\beta})$$

The source ρ is a symmetric 2-tensor given in terms of the stress energy tensor T by

$$\rho_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}trT, \quad \text{with} \quad trT \equiv g^{\lambda\mu}T_{\lambda\mu}$$

Due to the Bianchi identities the left hand side of the Einstein equations satisfies the identities, with ∇_α the covariant derivative in the metric g

$$\nabla_\alpha(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\ R) \doteq 0, \quad R \equiv g^{\lambda\mu}R_{\lambda\mu}$$

The stress energy tensor of the sources satisfies the conservation laws which make the equations compatible

$$\nabla_\alpha T^{\alpha\beta} = 0$$

In vacuum the stress energy tensor is identically zero. We will consider here only this case. The presence of sources brings up new problems specific to various types of sources.

The Einstein equations (in vacuum) are a geometric system, invariant by diffeomorphisms of V and the associated isometries of g. From the analyst point of view they constitute a system of second order quasilinear partial differential equations which is both undetermined (the characteristic determinant is identical to zero) and overdetermined (one cannot give arbitrarily Cauchy data).

3 Intrinsic Cauchy problem

Due to the geometric nature of Einstein's equations it is appropriate to consider a Cauchy problem also in geometric form. The definition follows.

An initial data set is a triple $(M, \bar{g}_0, K_0)$ where M is a 3 dimensional manifold, $\bar{g}_0$ a riemannian metric on M and K_0 a symmetric 2 tensor.

An extension of an initial data set is a spacetime (V, g) such that there exists an immersion $i : M \rightarrow M_0 \subset V$ with $\overset{*}{i} \bar{g}_0$ and $i^* K_0$ equal respectively to the metric induced by g on M_0 and the extrinsic curvature of M_0 as submanifold of (V, g).

We say that (V, g) is an einsteinian extension if g satisfies the Einstein (vacuum) equations on V.

A spacetime (V, g) is said to be globally hyperbolic if the set of timelike curves, between two arbitrary points is relatively compact in the Frechet topology of curves on V. This definition given by Leray 1952 has been shown by Geroch 1970 to be equivalent to the fact that (V, g) possesses a Cauchy surface, i.e. a spacelike submanifold M_0 such that each inextendible timelike or null curve cuts M_0 exactly once.

A development is a globally hyperbolic einsteinian extension.

4   3+1 splitting

To link geometry with analysis one performs a 3+1 splitting of the Einstein equations. We consider a manifold V of the type $M \times R$ (the support of a development will always be of this type). We denote by $x^i, \in i = 1, 2, 3$ local coordinates in M, we set $x^0 = t \in R$. We choose a moving frame on V such that at a point (x, t) its axes e_i coincide with the axis of the natural frame, tangent to $M_t \equiv M \times \{t\}$, and its axis e_0 is orthogonal to M_t, with associated coframe such that $\theta^0 = dt$. A generic lorentzian metric on V with the M_t 's spacelike reads in the associated coframe

$$g = -N^2 dt^2 + \bar{g}_{ij} \theta^i \theta^j, \quad \text{with} \quad \theta^i \equiv dx^i + \beta^i dt$$

The coefficients are time dependent geometric objects on M. The scalar N is called lapse, the space vector β is called shift, $\bar{g}$ is a riemannian metric. These elements are linked with the metric coefficients $g_{\alpha\beta}$ in the natural frame by the relations:

$$g_{ij} \equiv \bar{g}_{ij}, \quad N^2 = (-g^{00})^{-1}, \quad \beta^i \equiv N^2 g^{0i}$$

We denote by $\bar{\nabla}$ the covariant derivative in the metric $\bar{g}$. We have

$$\partial_i = \frac{\partial}{\partial x^i}, \quad \partial_0 = \partial_t - \beta^i \partial_i, \quad \text{with} \quad \partial_t = \frac{\partial}{\partial t}$$

and we denote

$$\hat{\partial}_0 = \frac{\partial}{\partial t} - L_\beta$$

with L_β the Lie derivative with respect to β, an operator wich maps a time dependent tensor field on M into another such tensor field.

We denote by K the extrinsic curvature of $M_t \equiv M \times \{t\}$ as submanifold of (V,g), i.e. we set:

$$K_{ij} = -\frac{1}{2N}\hat{\partial}_0 g_{ij}$$

A straightforward calculation (C-B 1956) gives the fundamental identities, written in the coframe $\theta^0 = dt, \theta^i = dx^i + \beta^i dt$, for the Ricci tensor of g :

$$R_{ij} \equiv \bar{R}_{ij} - \frac{\hat{\partial}_0 K_{ij}}{N} - 2K_{jh}K_i^h + K_{ij}K_h^h - \frac{\bar{\nabla}_j \partial_i N}{N}$$

$$R_{0i} \doteq N(-\bar{\nabla}_h k_i^h + \bar{\nabla}_i k_h^h)$$

$$R_{00} \doteq N(\partial_0 K_h^h - NK_{ij}K^{ij} + \bar{\nabla}^i \partial_i N)$$

5 Constraints and evolution.

Constraints

The following part of the Einstein equations do not contain second derivatives of g neither first derivatives of K transversal to the spacelike manifolds M_t. They are the constraints. They read, with $S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$,

Momentum constraint

$$C_i \doteq \frac{1}{N}R_{0i} \doteq -\bar{\nabla}_h K_i^h + \bar{\nabla}_i K_h^h = 0$$

Hamiltonian constraint

$$C_0 \doteq \frac{2}{N^2}S_{00} \equiv \bar{R} - K_j^i K_i^j + (K_h^h)^2 = 0$$

These constraints are transformed into a system of elliptic equations on each submanifold M_t, in particular on M_0 for $\bar{g}= g_0, K = K_0$, by the conformal method .

Evolution.

The equations

$$R_{ij} \equiv \bar{R}_{ij} - \frac{\hat{\partial}_0 K_{ij}}{N} - 2K_{jh}K_i^h + K_{ij}K_h^h - \frac{\bar{\nabla}_j \partial_i N}{N} = 0$$

together with the definition

$$\hat{\partial}_0 g_{ij} = -2NK_{ij}$$

determine the derivatives transversal to M_t of $\bar{g}$ and K when these tensors are known on M_t as well as the lapse N and shift β. It is natural to look at these equations as evolution equations determining $\bar{g}$ and K, while N and β, projections of the tangent to the time line respectively on the normal and the tangent space to M_t, are considered as gauge variables. This point of view is conforted by the following theorem (Anderson and York 1997, previously given for sources in C-B and Noutcheguéme 1988)

Theorem 1 *When $R_{ij} = 0$ the constraints satisfy a linear homogeneous first order symmetric hyperbolic system, they are satisfied if satisfied initially.*

Proof. When $R_{ij} = 0$ we have, in the privileged frame,

$$R = -N^2 R^{00}$$

hence

$$S^{00} = \frac{1}{2}R^{00} \quad \text{and} \quad R = -2N^2 S^{00} = 2S_0^0$$

and

$$S^{ij} = -\frac{1}{2}\bar{g}^{ij}R = -\bar{g}^{ij}S_0^0$$

the Bianchi identities give therefore a linear homogeneous system for S_0^i and S_0^0 with principal parts

$$N^{-2}\partial_0 S_0^i + \bar{g}^{ij}\partial_j S_0^0, \quad \text{and} \quad \partial_0 S_0^0 + \partial_i S_0^i$$

This system is symetrizable hyperbolic, it has a unique solution, zero if the initial values are zero. The characteristics, which determine the domain of dependence, are the light cone.

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6 Hyperbolicity non strict of $R_{ij} = 0$

An evolution part of Einstein equations should exhibit causal propagation, i.e. with domain of dependence determined by the light cone of the spacetime metric.

The equations $R_{ij} = 0$ are, when N and β are known, a second order differential system for g_{ij}. The hyperbolicity of a quasilinear system is defined through the linear differential operator obtained by replacing in the coefficients the unknown by given values. In our case for given N, β and g_{ij} the principal part of this operator acting on a symmetric 2-tensor γ_{ij} is

$$\frac{1}{2}(N^{-2}\partial_{00}^2 - g^{hk}\partial_{hk}^2)\gamma_{ij} + \partial^k\partial_j\gamma_{ik} + \partial^k\partial_i\gamma_{jk} - g^{hk}\partial_i\partial_j\gamma_{hk}$$

The characteristic matrix at a point of spacetime is the linear operator obtained by replacing the derivation ∂ by a covariant vector ξ. The characteristic determinant is the determinant of this linear operator. We take as independent unknown $\gamma_{12}, \gamma_{23}, \gamma_{31}, \gamma_{11}, \gamma_{22}, \gamma_{33}$ and consider the 6 equations $R_{ij} = 0$,same indices.

To simplify the writing we compute this matrix in a coframe orthonormal for the given spacetime metric (N, β, g_{ij}). We denote by (t, x, y, z) the components of ξ in such a coframe. The characteristic matrix $\mathcal{M}$ reads then (up to multiplication by 2):

$$\mathcal{M} \equiv \begin{matrix} t^2 - z^2 & zx & zy & 0 & 0 & -xy \\ xz & t^2 - x^2 & xy & -yz & 0 & 0 \\ yz & xy & t^2 - y^2 & 0 & -xz & 0 \\ 2xy & 0 & 2zx & t^2 - y^2 - z^2 & -x^2 & -x^2 \\ 2xy & 2yz & 0 & -y^2 & t^2 - x^2 - z^2 & -y^2 \\ 0 & 2yz & 2xz & -z^2 & -z^2 & t^2 - x^2 - y^2 \end{matrix}$$

The characteristic polynomial is the determinant of this matrix. It is found to be

$$Det\mathcal{M} = b^6 a^3, \quad \text{with} \quad b = t, \quad a = t^2 - x^2 - y^2 - z^2$$

The characteristic cone is the dual of the cone defined in the cotangent plane by annulation of the characteristic polynomial.For our system the characteristic cone splits into the light cone of the given spacetime metric and the normal to its space slice. Since these charateristics appear as multiple and the system is non diagonal it is not hyperbolic in the usual sense. We will prove the following theorem

Theorem 2 *When $N > 0$ and β are given, arbitrary, the system $R_{ij} = 0$ is a system hyperbolic non strict in the sense of Leray Ohya for g_{ij}, in the Gevrey class $\gamma = 2$, as long as g_{ij} is properly riemannian. If the Cauchy data as well as N and β are in such a Gevrey class the Cauchy problem has a local in time solution, with domain of dependence determined by the light cone.*

Proof.

The product $ab^2 \mathcal{M}^{-1}$, with $\mathcal{M}^{-1}$ the inverse of the characteristic matrix $\mathcal{M}$ is computed to be:

$$
\begin{matrix}
t^2 - x^2 - y^2 & -zx & -zy & 0 & 0 & xy \\
-zx & t^2 - y^2 - z^2 & -xy & zy & 0 & 0 \\
-zy & -xy & t^2 - x^2 - z^2 & 0 & zx & 0 \\
-2xy & 0 & -2zx & t^2 - x^2 & x^2 & x^2 \\
-2xy & -2zy & 0 & y^2 & t^2 - y^2 & y^2 \\
0 & -2zy & -2zx & z^2 & z^2 & t^2 - z^2
\end{matrix}
$$

We see that the elements of the matrix $ab^2 \mathcal{M}^{-1}$ are polynomials in x, y, z. The product of this matrix by $\mathcal{M}$ is a diagonal matrix with elements ab^2 in the diagonal. Consider now the differential operator R_{ij} acting on g_{ij}. Multiply it on the left by the differential operator defined by replacing in $ab^2 \mathcal{M}^{-1}$ the variables x, y, z by the derivatives $\partial_1, \partial_2, \partial_3$. The resulting operator is quasi diagonal with principal operator $\partial^\lambda \partial_\lambda \partial_0^2$. It is the product of two strictly hyperbolic operators, $\partial^\lambda \partial_\lambda \partial_0$ and ∂_0. The result follows from the Leray-Ohya general theory.

7 Hyperbolic non strict 4th order system.

Lemma 3 *The following combination of derivatives of components of the Ricci tensor of an arbitrary spacetime :*

$$\Lambda_{ij} \equiv \hat{\partial}_0 \hat{\partial}_0 R_{ij} - \hat{\partial}_0 \bar{\nabla}_{(i} R_{j)0} + \bar{\nabla}_j \partial_i R_{00}$$

reads, when g is known, as a third order quasi diagonal hyperbolic system for the extrinsic curvature K_{ij}.

$$
\begin{aligned}
\Lambda_{ij} \equiv{} & \hat{\partial}_0 \mathcal{D} K_{ij} + \hat{\partial}_0 \hat{\partial}_0 (H K_{ij} - 2 K_{im} K_j^m) - \hat{\partial}_0 \hat{\partial}_0 (N^{-1} \bar{\nabla}_j \partial_i N) + \\
& \hat{\partial}_0 (-\bar{\nabla}_{(i} (K_{j)h} \partial^h N) - 2N \bar{R}_{::ijm}^h K_h^m - N \bar{R}_{m(i} K_{j)}^m + H \bar{\nabla}_j \partial_i N) + \\
& \bar{\nabla}_j \partial_i (N \bar{\Delta} N - N^2 K.K) + \mathcal{C}_{ij}
\end{aligned}
$$

with

$$\mathcal{D}K_{ij} \equiv -\hat{\partial}_0(N^{-1}\hat{\partial}_0 K_{ij}) + \bar{\nabla}^h\bar{\nabla}_h(NK_{ij}), \quad \bar{\Delta} = \bar{\nabla}_h\bar{\nabla}^h, \quad H \equiv K_h^h$$

and

$$\mathcal{C}_{ij} \equiv \bar{\nabla}_j\partial_i(N\partial_0 H) - \hat{\partial}_0(N\bar{\nabla}_j\partial_i H)$$

Proof. A straightforward computation shows that $\mathcal{C}_{ij}$ contains terms of at most second order in K (and also in N) and first order in $\bar{g}$ (replace $\hat{\partial}_0 g_{ij}$ by $-2NK_{ij}$). The other terms of Λ_{ij}, except for $\hat{\partial}_0\mathcal{D}K_{ij}$ are second order in K. All terms of Λ_{ij} are at most second order in $\bar{g}$ except for third order terms appearing through $\bar{\nabla}_j\partial_i(N\bar{\Delta}N)$. Because of these terms the system for $\bar{g}$ and K given by

$$\Lambda_{ij} = 0 \tag{1}$$

and

$$\hat{\partial}_0 g_{ij} = -2NK_{ij} \tag{2}$$

is not quasi diagonal. It is not hyperbolic in the usual sense of Leray, we will prove the following theorem.

Theorem 4 *The system (1), (2) with unknown $\bar{g}$, K is for any choice of lapse N and shift β equivalent to a system hyperbolic non strict in the sense of Leray-Ohya with local existence of solutions in Gevrey classes $\gamma = 2$ and domain of dependence determined by the light cone.*

Proof. Replace in the equations $\Lambda_{ij} = 0$ the tensor K by $-(2N)^{-1}\hat{\partial}_0\bar{g}$: this gives a quasi diagonal system for $\bar{g}$, but with principal operator $(\partial_0)^2\partial^\lambda\partial_\lambda$. The result follows immediately from the Leray-Ohya theory.

The system for $\bar{g}$, K can be turned into a *hyperbolic system* by a *gauge choice* as follows.

Theorem 5 *Suppose that N satisfies the wave equation.*

$$N^{-2}\partial_0\partial_0 N - \bar{\Delta}N = f \tag{3}$$

with f an arbitrarily given smooth function on $M \times R$. The system (1),(2),(3), called $\mathcal{S}$, is equivalent to a hyperbolic Leray system for $\bar{g}, K, N$, for arbitrary shift.

Proof. We use the wave equation (3) to reduce the terms $-\hat{\partial}_{00}^2(N^{-1}\bar{\nabla}_j\partial_i N)+$
$\bar{\nabla}_j\partial_i(N\bar{\Delta}N)$ in Λ_{ij} to terms of third order in N, second order in g and K.

We replace the equation (3) by the also hyperbolic equation

$$\hat{\partial}_0(N^{-2}\partial_0\partial_0 N - \bar{\Delta}N) = \partial_0 f \tag{4}$$

and replace $\hat{\partial}_0\bar{g}$ by $-2NK$ wherever it appears. Then the equation (4) is third order in N, while first order in $\bar{g}$ and K. We call $\mathcal{S}'$ the system thus modified.

We give to the equations and unknowns the Leray-Volevic indices:

$$m(1) = 0, \quad m(2) = 2, \quad m(3) = 1 \tag{5}$$

$$n(g) = n(K) = 3, \quad n(N) = 4 \tag{6}$$

The principal matrix of the system $\mathcal{S}'$ is then diagonal with elements the hyperbolic operators $\partial_0\partial^\alpha\partial_\alpha$ or ∂_0.

If the equation (3) is satisfied on the initial submanifold as well as $\mathcal{S}'$, the equation (3) and the system $\mathcal{S}$ are satisfied.

Remark 6 *The system $\Lambda_{ij} = 0$ has the additionnal property to satisfy a polarized null condition, that is the quadratic form defined by the second derivative of A at some given metric g, $\Lambda"_{ij}(g)(\gamma,\gamma)$, vanishes when $\gamma = \ell\otimes\ell$ with ℓ a null vector for the spacetime metric g such that γ is in the kernel of the first derivative of the Ricci tensor of spacetime at g (C-B 2000).*

8   An hyperbolic second order system

A variety of hyperbolic evolution systems for Einstein equations have been obtained, with a speed greatly increasing in recent years, by replacing the trivial gauge choice (which is the data of N and β on V) by more elaborate ones, together with combining the evolution equations with the constraints. The hope in changing the gauge is to find systems either better suited to the study of global existence problems, or more stable under numerical codes. We give some references in the bibliography. We will return below to the original gauge choice (C-B 1952) in the perspective of conditions on the lapse and the shift.

The following identity was already known by De Donder, Lanczos and Darmois. It splits the Ricci tensor with components $R_{\alpha\beta}$ in the natural

frame into a quasi linear quasi diagonal wave operator and 'gauge' terms, as follows:

$$R_{\alpha\beta} \doteq R_{\alpha\beta}^{(h)} + \frac{1}{2}(g_{\beta\lambda}\frac{\partial}{\partial x^\alpha}F^\lambda + g_{\alpha\lambda}\frac{\partial}{\partial x^\beta}F^\lambda)$$

with

$$R_{\alpha\beta}^{(h)} \doteq -\frac{1}{2}g^{\lambda\mu}\frac{\partial^2}{\partial x^\lambda \partial x^\mu}g_{\alpha\beta} + H_{\alpha\beta} = 0$$

where $H_{\alpha\beta}$ is a quadratic form in first derivatives of g with coefficients polynomials in g and its contravariant associate.

The F^λ are given by

$$F^\lambda \equiv g^{\alpha\beta}\Gamma_{\alpha\beta}^\lambda \equiv \nabla^\alpha\nabla_\alpha x^{(\lambda)}$$

They are non tensorial quantities, result of the action of the wave operator of g on the coordinate functions. For this reason their vanishing is called 'harmonicity condition'.

The contravariant components of the Ricci tensor admit an analogous splitting, namely:

$$R^{\alpha\beta} \doteq R_{(h)}^{\alpha\beta} + \frac{1}{2}(g^{\alpha\lambda}\frac{\partial}{\partial x^\lambda}F^\beta + g^{\beta\lambda}\frac{\partial}{\partial x^\lambda}F^\alpha)$$

with

$$R^{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}\frac{\partial^2}{\partial x^\lambda \partial x^\mu}g^{\alpha\beta} + K^{\alpha\beta}$$

Let us denote by $R_{\alpha\beta}^{(\theta)}$ the components of the Ricci tensor in the previous frame $\theta^0 = dt$, $\theta^i = dx^i + \beta^i dt$, to distinguish them from the components in the natural frame now denoted $R_{\alpha\beta}$. These components are linked by the relations:

$$R_{ij}^{(\theta)} = R_{\alpha\beta}\frac{\partial(dx^\alpha)}{\partial\theta^i}\frac{\partial(dx^\beta)}{\partial\theta^j} = R_{ij},$$

$$R^{00} = R_{(\theta)}^{\alpha\beta}\frac{\partial(dt)}{\partial\theta^\alpha}\frac{\partial(dt)}{\partial\theta^\beta} = R_{(\theta)}^{00}$$

$$R_{(\theta)}^{0i} = R^{\alpha\beta}\frac{\partial(\theta^0)}{\partial(dx^\alpha)}\frac{\partial(\theta^i)}{\partial(dx^\beta)} = R^{00} + R^{0j}$$

The equations $R_{ij}^{(h)} = 0$ are a quasidiagonal second order system for g_{ij} when N and β are known. The equations $R_{(h)}^{00} = 0$ and $R_{(h)}^{0i}$ are quasilinear wave

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equations for N and β when the $g'_{ij}s$ are known: we interpret these equations as gauge conditions. The set of all these equations constitute a quasidiagonal second order system for g_{ij}, N and β, hyperbolic and causal as long as $N > 0$ and $\bar{g}$ is properly riemannian.

The Bianchi identities show that for a solution of these equations the quantities F^λ satisfy a linear homogeneous quasidiagonal hyperbolic and causal system. Its initial data can be made zero by choice of initial coordinates if and only if the geometric initial data $\bar{g}, K$ satisfy the constraints. A solution of our hyperbolic system satisfies then the full Einstein equations.

The following local existence and uniqueness theorem improves the differentiability obtained in the original theorem of C-B 1952 who used C^k spaces and a constructive (parametrix) method. The improvement to Sobolev spaces (one can endow M with a given smooth riemannian metric to define those spaces) with $s \geq 4$ for existence and $s \geq 5$ for geometric uniqueness is given in C-B 1968 using Leray's results. The improvement given in the theorem was sugested by Hawking and Ellis 1973, proved by semigroup methods by Hughes, Kato and Marsden 1978 and by energy methods by C-B, Christodoulou and Francaviglia 1979. The other hyperbolic systems constructed in the past twenty years did not lead, up to now, to further improvement on the regularity required of the Cauchy data. Such an improvement would be an important step.

Theorem 7 *Given an initial data set, $\bar{g}_0, K_0 \in H_s^{local}, H_{s-1}^{local}$ satisfying the constraints, there exists an einsteinian extension if $s \geq 3$.*

The question of uniqueness is a geometrical problem. It is in general easy to prove that the solution is unique in the chosen gauge, for instance in the harmonic gauge recalled above. But two isometric spacetimes must be considered as identical. The following theorem (C-B and Geroch) gives this geometric uniqueness (maximal means inextendible).

Theorem 8 *The development of an initial data set is unique up to isometries in the class of maximal developments if $s \geq 4$. The domain of dependence is determined by the light cone of the spacetime metric.*

The proof of C-B and Geroch considers smooth data and developments, the refined result is due to Chrusciel 1996. The geometric uniqueness in the case $s = 3$, even the local one, is still an open problem.

9 Bianchi equations.

The Riemann tensor satisfies the identities

$$\nabla_\alpha R_{\beta\gamma,\lambda\mu} + \nabla_\beta R_{\gamma\alpha,\lambda\mu} + \nabla_\gamma R_{\alpha\beta,\lambda\mu} \equiv 0 \tag{7}$$

it holds therefore that, modulo the symmetries of the Riemann tensor

$$\nabla_\alpha R^\alpha_{..\mu,\beta\gamma} \equiv \nabla_\beta R_{\gamma\mu} - \nabla_\gamma R_{\beta\mu} \tag{8}$$

hence if the Ricci tensor $R_{\alpha\beta}$ satisfies the vacuum Einstein equations

$$R_{\alpha\beta} = 0 \tag{9}$$

it holds that

$$\nabla_\alpha R^\alpha_{..\mu,\beta\gamma} = 0. \tag{10}$$

The system (8), (11) splits as the Eintein equations into constraints, containing no time derivatives of curvature, namely in the frame used in the 3+1 splitting:

$$\nabla_i R_{jk,\lambda\mu} + \nabla_k R_{ij,\lambda\mu} + \nabla_j R_{ki,\lambda\mu} \equiv 0 \tag{11}$$

$$\nabla_\alpha R^\alpha_{..0,\beta\gamma} = 0. \tag{12}$$

and an evolution system

$$\nabla_0 R_{hk,\lambda\mu} + \nabla_k R_{0h,\lambda\mu} - \nabla_h R_{0k,\lambda\mu} = 0 \tag{13}$$

$$\nabla_0 R^0_{:::i,\lambda\mu} + \nabla_h R^h_{:::i,\lambda\mu} = 0 \tag{14}$$

This system has a principal matrix consisting of 6 identical 6 by 6 blocks around the diagonal, obtained by fixing a pair λ, μ, $\lambda < \mu$. Each block is symmetrizable through the metric $\bar{g}$, and hyperbolic if $\bar{g}$ is properly Riemannian and $N > 0$ because the principal matrix M^0 for the derivatives ∂_0 was, up to product by N^{-1}, the unit matrix and the derivatives ∂_h do not contain $\partial/\partial t$.

The Bel-Robinson energy is the energy associated to this symmetric hyperbolic system.

Remark 9 *Following Bel one can introduce two pairs of gravitational "electric" and "magnetic" space tensors associated with the 3+1 splitting of the spacetime and the double two-form Riemann(g) :*

$$N^2 E_{ij} \equiv R_{0i,0j}, \quad D_{ij} \equiv \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hk,lm}$$

$$N H_{ij} \equiv \frac{1}{2}\eta_{ihk}R^{hk}_{:::::,0j}, \quad N B_{ji} \equiv \frac{1}{2}\eta_{ihk}A^{:::::hk}_{0j,}$$

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where η_{ijk} is the volume form of $\bar{g}$. The principal part of the evolution system ressemble then to the Maxwell equations, but contains an additional non principal part. Its explicit expression is given in Anderson,C-B and York 1997.

The Bianchi equations do not tell the whole story since they contain the spacetime metric g, which itself depends on the Riemann tensor.

A possibility to obtain a symmetric evolution system (Friedrich 1996 with the Weyl tensor) for both g and $Riemann(g)$ (Anderson, C-B and York 1997) is to introduce again the auxiliary unknown K and use $3+1$ identities, involving now not only the Ricci tensor but also the Riemann tensor. One can then obtain a symmetric first order hyperbolic system for K and $\bar{\Gamma}$, the space metric connection, modulo a choice of gauge, namely the integrated form of the harmonic time-slicing condition used before (C-B and Ruggeri 1983). The energy associated to this system has unfortunately no clear geometrical meaning. Determination of the metric from the Riemann tensor through elliptic equations seems more promising for the solution of global problems (see Christodoulou and Klainerman 1989, Andersson and Moncrief, in preparation)

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